A Proof of The Existence of An Algebraic Closure Using Ultraproducts

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June 17, 2023

One of the exercises in [1] (Chapter 3, Problem 9) was to prove the existence of an algebraic closure using ultraproducts. It was quite interesting as an example of the application of model theory, so I will make a note of it.

Problem 1.

Prove that all fields have an algebraic closure.

Firstly, we prepare a theorem.

Theorem 2.(Łos)

Fix a language $\mathcal{L}$. Let I be a nonempty set, $\mathcal{U}$ be an ultrafilter on I, and $\{\mathfrak{A}_i\}_{i\in I}$ be a family of $\mathcal{L}$ structures indexed by I. For an arbitrary formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in \prod_{i\in I} \mathfrak{A}_i$,

$$\prod_{i\in I} \mathfrak{A}_i/\mathcal{U} \models \varphi([a_1], \ldots, [a_n]) \iff \|\varphi(a_1, \ldots, a_n)\| \in \mathcal{U}.$$

Here, $a_k = ({a_k}^i)_{i\in I}$ and $\|\varphi(a_1, \ldots, a_n)\| = \{i \in I \mid \mathfrak{A}_{i|\mathfrak{A}_i|} \models \varphi({a_1}^i, \ldots, {a_n}^i)\}$.

Proof. omitted. □

Using this theorem, we prove the existence of an algebraic closure.

Theorem 3.(The existence of an algebraic closure)

Any field K has an algebraic closure.

Proof. Let $I = K[x] \setminus K$. Firstly, we construct a large field including K using an ultrafilter on I. For $p \in I$, let K_p be a smallest splitting field of p, and

$$J_p = \{q \in I \mid p \text{ is factored into a product of linear factors in } K_q\}.$$

Since

$$p_1 \cdots p_n \in J_{p_1} \cap \cdots \cap J_{p_n},$$

$\{J_p \mid p \in I\}$ has finite intersection property and can be extended to an ultrafilter $\mathcal{U}$ on I. Let $L := \prod_{p \in I} K_p/\mathcal{U}$. Since the axioms of a field are elementary formulae, Theorem 2 implies that L is also a field.

$$K \ni a \mapsto [(a)_{p \in I}] \in L$$

is an embedding and L/K. Let $\varphi \equiv \exists a \exists a_1 \ldots \exists a_n \forall x (f(x) = a(x - a_1) \cdots (x - a_n))$ for an arbitrary polinomial $f \in I$ of degree n. If $p \in J_f$, then $K_p \vDash \varphi$. Hence $\|\varphi\| \supseteq J_f$ and $\|\varphi\| \in \mathcal{U}$. This and Theorem 2 imply that $L \vDash \varphi$. Therefore L has all roots of arbitrary non-constant polynomials on K.

Next, we prove that the integral closure $\overline{K}$ of K in L is an algebraic closure of K. $\overline{K}/K$ is an algebraic extension by definition, so it is enough to see that $\overline{K}$ is algebraically closed. Let

$$f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \overline{K}[x] \setminus \overline{K}.$$

$f \in (K(a_0, \ldots, a_n))[x]$. Let M denote the smallest splitting of f. Since $M/K(a_0, \ldots, a_n)$ and $K(a_0, \ldots, a_n)/K$ are algebraic extensions, M/K is also an algebraic extension. Therefore the roots of f are algebraic elements over K in L, which means they are in $\overline{K}$. $\square$

References

[1] Kazuyuki Tanaka. 数学基礎論序説 —数の体系への論理的アプローチ—. 2019